ISRAEL JOURNAL OF MATHEMATICS 145 (2005), 83–91

OPERATORS THAT ADMIT A MOMENT SEQUENCE

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ABSTRACT

We consider the question of which operators on Hilbert space admit a moment sequence, and show, in particular, that if T is any operator in $\mathcal{L}(\mathcal{H})$ that can be written as $T = N + K$, where N is normal and K is compact, and $\mathcal{M} \subset \mathcal{H}$ is any invariant subspace for T, then the restriction $T|_{\mathcal{M}}$ admits such a sequence.

Received January 27, 2004

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. The (closed) ideal of all compact operators in $\mathcal{L}(\mathcal{H})$ will be denoted by $\mathbf{K}$, and the quotient (Calkin) map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ by π. We write, as usual, $\mathbb{N}[\mathbb{N}_0]$ for the set of positive [nonnegative] integers, and for T in $\mathcal{L}(\mathcal{H})$ we write $\sigma(T)$ for the spectrum of T and $\sigma_e(T) = \sigma(\pi(T))$. The set of all operators T in $\mathcal{L}(\mathcal{H})$ that can be written in the form $T = N + K$ where N is normal and $K \in \mathbf{K}$ will be denoted by $(\mathbf{N}+\mathbf{K})(\mathcal{H})$ or, more simply, by $(\mathbf{N}+\mathbf{K})$.

Definition 1.1: An operator T in $\mathcal{L}(\mathcal{H})$ will be said to **admit a moment sequence** if there exist nonzero vectors u_0, v_0 in $\mathcal{H}$ and a Borel measure (as opposed to a signed or complex measure) μ_{u_0,v_0} supported on $\sigma(T)$ such that

$$(1) \qquad (T^n u_0, v_0) = \int_{\sigma(T)} \lambda^n d\mu_{u_0,v_0}, \quad n \in \mathbb{N}_0.$$

Moreover, if (1) holds, then the pair (u_0, v_0) is said to **induce** a moment sequence for T.

This note addresses the following problem.

PROBLEM 1.2: *Which operators in $\mathcal{L}(\mathcal{H})$ admit a moment sequence?*

The interest in this question arises because of the following nice theorem of Atzmon–Godefroy [3].

THEOREM 1.3: *Let $\mathcal{X}$ be a real Banach space that has an equivalent Gateaux smooth norm (which is true, in particular, if $\mathcal{X}$ is either reflexive or separable), and let T be a bounded linear operator on $\mathcal{X}$ such that T admits a moment sequence (with associated measure supported on $\mathbb{R}$). Then T has a nontrivial invariant subspace.*

One would hope that the analog of this theorem for operators in $\mathcal{L}(\mathcal{H})$ is true, but, in any case, a natural question to ask is which classes of operators in $\mathcal{L}(\mathcal{H})$ admit a moment sequence? Our main result is the following.

THEOREM 1.4: *Every operator T in $\mathcal{L}(\mathcal{H})$ that can be written as a restriction $T = \widetilde{T}|_{\mathcal{H}}$, where $\widetilde{T} \in (\mathbf{N}+\mathbf{K})(\widetilde{\mathcal{K}})$ for some $\widetilde{\mathcal{K}} \supset \mathcal{H}$ and $\widetilde{T}\mathcal{H} \subset \mathcal{H}$, admits a moment sequence.*

The proof of Theorem 1.4 will be given in Section 2, but first we note that Scott Brown in [6] proved a special case of Theorem 1.4; namely, if $T \in (\mathbf{N}+\mathbf{K})(\mathcal{H})$ and $\sigma(T)$ is thin, then T admits a moment sequence.

We will need the following elementary facts.

PROPOSITION 1.5: *The set of all operators in $\mathcal{L}(\mathcal{H})$ which have a moment sequence is selfadjoint and closed under multiplication by scalars. Moreover, every T in $\mathcal{L}(\mathcal{H})$ which has a nontrivial invariant subspace admits a moment sequence.*

Proof: The first statement is trivial, and for the second, let u_0 be a nonzero vector in $\mathcal{M}$ and v_0 be a nonzero vector in $\mathcal{M}^{\perp}$. Then (1) is obviously satisfied with μ_{u_0,v_0} the Borel measure that is identically zero on Borel subsets of $\sigma(T)$. ∎

PROPOSITION 1.6: *Suppose $T \in \mathcal{L}(\mathcal{H})$, $\{u_k\}$ and $\{v_k\}$ are sequences in $\mathcal{H}$ converging in norm to nonzero vectors u_0 and v_0, respectively, and for every $k \in \mathbb{N}$ there exists a Borel measure μ_{u_k,v_k} supported on $\sigma(T)$ such that*

$$(T^n u_k, v_k) = \int \lambda^n d\mu_{u_k,v_k}, \quad k \in \mathbb{N},\ n \in \mathbb{N}_0.$$

Then there exists a Borel measure μ_{u_0,v_0} supported on $\sigma(T)$ such that

$$(T^n u_0, v_0) = \int \lambda^n d\mu_{u_0,v_0}, \quad n \in \mathbb{N}_0.$$

Proof: Let $C(\sigma(T))$ denote, as usual, the Banach space of continuous complex valued functions on $\sigma(T)$ under the sup norm. Then the measures μ_{u_k,v_k} belong to $C(\sigma(T))^*$, and since

$$\mu_{u_k,v_k}(\sigma(T)) = (u_k, v_k) \leq \|u_k\|\|v_k\|, \quad k \in \mathbb{N}_0,$$

and the sequences $\{u_k\}$ and $\{v_k\}$ converge, the sequence $\{\|\mu_{u_k,v_k}\|\}$ is bounded. Thus there is a subsequence $\{\mu_{u_{k_m},v_{k_m}}\}_{m\in\mathbb{N}}$ that is weak* convergent to some (nonnegative) Borel measure μ_0 (which is obviously supported on $\sigma(T)$), and consequently

$$(T^n u_0, v_0) = \int_{\sigma(T)} \lambda^n d\mu_0, \quad n \in \mathbb{N}_0,$$

so $\mu_{u_0,v_0} = \mu_0$. ∎

2. The main result

The proof of Theorem 1.4 depends on [7, Theorem 2.1], which we now state for the reader's convenience in the abbreviated form in which it will be used.

THEOREM 2.1: *Let $\mathcal{A}$ be a unital, norm-separable, norm-closed subalgebra of $\mathcal{L}(\mathcal{H})$ and let $Q \in \mathcal{L}(\mathcal{H})$ be such that $0 \leq Q \leq 1$ and $Q(1-Q) \neq 0$. Then the following statements are equivalent:*

(a) *there exists a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of (orthogonal) projections in $\mathcal{L}(\mathcal{H})$ such that*

$$\|(1-Q_n)AQ_n\| \longrightarrow 0, \quad A \in \mathcal{A}, \tag{2}$$

and $\{Q_n\}$ converges to Q in the weak operator topology (WOT).

(b) *With $\mathcal{M} := (Q\mathcal{H})^-(\neq (0))$ and $\mathcal{N} := ((1-Q)Q\mathcal{H})^-(\neq (0))$, there exist linear multiplicative mappings $\varphi\colon \mathcal{A} \to \mathcal{L}(\mathcal{M})$ and $\theta\colon \pi(\mathcal{A}) \to \mathcal{L}(\mathcal{N})$, with θ completely contractive and $\theta(\pi(1_{\mathcal{H}})) = 1_{\mathcal{N}}$, uniquely determined by the relations*

$$Q^{1/2}\varphi(A) = AQ^{1/2}|\mathcal{M}, \quad A \in \mathcal{A}, \tag{3}$$

and

$$(1-Q)^{1/2}\varphi(A) = \theta(\pi(A))(1-Q)^{1/2}|\mathcal{M}, \quad A \in \mathcal{A}. \tag{4}$$

We first obtain a corollary of Theorem 2.1 that is fundamental to our efforts. For information about quasitriangular operators, see Section 3 below.

THEOREM 2.2: *Suppose $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{A} = \mathcal{A}_T$ is the unital norm-closed algebra generated by T (and $1_{\mathcal{H}}$). Suppose also that $\{Q_n\}$ is a sequence of projections in $\mathcal{L}(\mathcal{H})$ satisfying (2) and converging in the weak operator topology (WOT) to an operator Q such that $Q(1-Q) \neq 0$. Then there exists a unital *-algebra homomorphism ρ from the C^*-algebra $C^*(\pi(\mathcal{A}))$ generated by the algebra $\pi(\mathcal{A})$ into some $\mathcal{L}(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space satisfying $\mathcal{K} \supset \mathcal{N} := \{Q(1-Q)\mathcal{H}\}^-$, such that*

$$(A(Q^{3/2}x), (1-Q)Q^{1/2}x) = (\rho(\pi(A)(1-Q)^{1/2}Qx, (1-Q)^{1/2}Qx), \quad A \in \mathcal{A}, \tag{5}$$

for all x in $\mathcal{H}$ such that $Q(1-Q)x \neq 0$. In particular, if T is quasitriangular and has no nontrivial invariant subspace, then such Q, ρ, and $\mathcal{K}$ with the above properties (including (5)) exist.

Proof: We show first that if T is quasitriangular and has no nontrivial invariant subspace, then such a sequence $\{Q_n\}$ and Q satisfying $Q(1-Q) \neq 0$ exist.

Let $\{P_n\}$ be an increasing sequence of finite-rank projections converging in the strong operator topology (SOT) to $1_{\mathcal{H}}$ such that

$$\|P_nTP_n - TP_n\| \longrightarrow 0. \tag{6}$$

We next apply the Aronszajn–Smith procedure (cf. [8, Chapter 4]) to the operator $P_nTP_n|\operatorname{range} P_n$ to get an invariant projection $Q_n \leq P_n$ such that

$$Q_n(P_nTP_n)Q_n = (P_nTP_n)Q_n \tag{7}$$

and $1/4 \leq \tau(Q_n) \leq 3/4$ for all $n \in \mathbb{N}$, where $\tau(X) := \frac{1}{2}(Xs_0, s_0) + \frac{1}{2}(Xt_0, t_0)$ for orthogonal unit vectors s_0 and t_0 in $\mathcal{H}$. Thus we have from (7) that

$$Q_nTQ_n = P_nTQ_n, \quad n \in \mathbb{N}, \tag{8}$$

and from (6) we see that $\|P_nTQ_n - TQ_n\| \to 0$. This, together with (8), gives

$$\|Q_nTQ_n - TQ_n\| \longrightarrow 0, \tag{9}$$

and an easy induction argument on the degree of the polynomial p shows that, for every polynomial p,

$$\|Q_np(T)Q_n - p(T)Q_n\| \longrightarrow 0. \tag{10}$$

By taking limits in (10) we get

$$\|Q_nAQ_n - AQ_n\| \longrightarrow 0, \quad A \in \mathcal{A}. \tag{11}$$

Furthermore, some subsequence $\{Q_{n_k}\}_{k\in\mathbb{N}}$ converges in the WOT, say to Q (satisfying $0 \leq Q \leq 1$), and one knows (cf. [8, Theorem 4.10]) that both spaces $\ker(Q)^\perp$ and $\ker(1-Q)$ are (perhaps trivial) invariant subspaces for T. Moreover, $1/4 \leq \tau(Q) \leq 3/4$, so $0 \neq Q \neq 1$. This gives $(\ker Q)^\perp \neq (0)$ and $\ker(1-Q) \neq \mathcal{H}$. Thus $1 - Q \neq 0$, and if $Q(1-Q) = 0$, then $\ker Q \neq (0)$ and $(\ker Q)^\perp \neq \mathcal{H}$. This gives that $(\ker Q)^\perp$ is a nontrivial invariant subspace for T, contrary to the hypothesis. Thus we may suppose that $Q(1-Q) \neq 0$. This shows that if T is quasitriangular and has no nontrivial invariant subspace, then such a sequence $\{Q_n\}$ and WOT-limit Q exist, so we continue with the general case.

We now apply Theorem 2.1 to the pair $(\mathcal{A}, Q)$, and note that (a) is valid by (11). Thus, by (b), with $\mathcal{M}$ as defined there, there exist mappings $\varphi\colon \mathcal{A} \to \mathcal{L}(\mathcal{M})$ and $\theta\colon \pi(A) \to \mathcal{L}(\mathcal{N})$ with the prescribed properties, including (3) and (4).

(Note that $\mathcal{N} \subset \mathcal{M}$ and that from (3) and (4) we get that $\varphi(1_{\mathcal{H}}) = 1_{\mathcal{M}}$ and that $\theta(\pi(1_{\mathcal{H}})) = 1_{\mathcal{N}}$.) From (3) we get immediately that

$$(12) \qquad (1-Q)^{1/2}AQ^{1/2}|_{\mathcal{M}} = (1-Q)^{1/2}Q^{1/2}\varphi(A), \quad A \in \mathcal{A}$$

and from (4) we obtain that

$$(13) \qquad Q^{1/2}\theta(\pi(A))(1-Q)^{1/2}|_{\mathcal{M}} = Q^{1/2}(1-Q)^{1/2}\varphi(A), \quad A \in \mathcal{A},$$

so putting together (12) and (13) we get

$$(14) \qquad Q^{1/2}\theta(\pi(A))(1-Q)^{1/2}|_{\mathcal{M}} = (1-Q)^{1/2}AQ^{1/2}|_{\mathcal{M}}, \quad A \in \mathcal{A}.$$

Since θ: $\pi(\mathcal{A}) \to \mathcal{L}(\mathcal{N})$ is completely contractive and $\theta(1_{\mathcal{L}(\mathcal{H})/\mathbf{K}}) = 1_{\mathcal{N}}$, by the Arveson extension theorem [2] there exists a completely positive map $\widetilde{\theta}$: $C^*(\pi(\mathcal{A})) \to \mathcal{L}(\mathcal{N})$ such that $\widetilde{\theta}|\pi(\mathcal{A}) = \theta$. Moreover, we can apply the Stinespring dilation theorem [9] to $\widetilde{\theta}$ to obtain a Hilbert space $\mathcal{K} \supset \mathcal{N}$ and a unital *-homomorphism ρ: $C^*(\pi(\mathcal{A})) \to \mathcal{L}(\mathcal{K})$ such that

$$\widetilde{\theta}(C) = P_{\mathcal{N}}\rho(C)|_{\mathcal{N}}, \quad C \in C^*(\pi(\mathcal{A})),$$

and thus that

$$(15) \qquad \theta(\pi(A)) = P_{\mathcal{N}}\rho(\pi(A))|_{\mathcal{N}}, \quad A \in \mathcal{A},$$

where $P_{\mathcal{N}}$ is the projection of $\mathcal{K}$ onto $\mathcal{N}$. Next suppose $x \in \mathcal{H}$ is any vector such that $(1-Q)Qx \neq 0$ (which yields $Q^{3/2}x \neq 0$ also). Thus we get, by applying (14) and then (15),

$$\begin{aligned}(AQ^{3/2}x, (1-Q)Q^{1/2}x) &= (Q^{1/2}\widetilde{\theta}(\pi(A))(1-Q)^{1/2}Qx, (1-Q)^{1/2}Q^{1/2}x)\\ &= (P_{\mathcal{N}}(\rho(\pi(A))|_{\mathcal{N}}(1-Q)^{1/2}Qx, (1-Q)^{1/2}Qx)\\ &= (\rho(\pi(A))((1-Q)^{1/2}Qx), ((1-Q)^{1/2}Qx)), \quad A \in \mathcal{A},\end{aligned}$$

and the proof is complete. ∎

Proof of Theorem 1.4: Let $T \in \mathcal{L}(\mathcal{H})$ and $\widetilde{T} \in \mathcal{L}(\widetilde{\mathcal{K}})$ (with $\widetilde{\mathcal{K}} \supset \mathcal{H}$) be as in the statement of the theorem, and write $\widetilde{T} = \widetilde{N} + \widetilde{K}$, when $\widetilde{N}$ is normal and $\widetilde{K}$ is compact. Observe next, by Proposition 1.5 and Theorem 3.1 below, that without loss of generality we may suppose that $T(= \widetilde{T}|_{\mathcal{H}})$ is quasitriangular and that T has no nontrivial invariant subspace. Let also the sequence $\{Q_n\} \subset \mathcal{L}(\mathcal{H})$ with WOT-limit Q satisfying (2) and $Q(1-Q) \neq 0$ be given by Theorem 2.2. Next we define $\widetilde{Q}_n$ and $\widetilde{Q}$ in $\mathcal{L}(\widetilde{\mathcal{K}})$ by

$$\widetilde{Q}_n = Q_n \oplus 0_{\widetilde{\mathcal{K}}\ominus\mathcal{H}}, \quad n \in \mathbb{N},$$

and

$$\widetilde{Q} = Q \oplus 0_{\widetilde{\mathcal{K}}\ominus\mathcal{H}}.$$

An easy matricial calculation with 2×2 operator matrices shows that

$$\|\widetilde{Q}_n\widetilde{A}\widetilde{Q}_n - \widetilde{A}\widetilde{Q}_n\| \longrightarrow 0, \quad \widetilde{A} \in \widetilde{\mathcal{A}},$$

where $\widetilde{\mathcal{A}}$ is the unital norm-closed algebra generated by $\widetilde{T}$, and that $\widetilde{Q}(1-\widetilde{Q}) \neq 0$. Thus Theorem 2.2 can be applied to $\widetilde{Q}$ and $\widetilde{\mathcal{A}}$ to obtain a unital *-algebra homomorphism $\widetilde{\rho}$ of $C^*(\pi(\widetilde{\mathcal{A}}))$ into a Hilbert space $\mathcal{K}$ such that (5) is valid. Since range $(1-\widetilde{Q})$ = range$(1-Q)Q \subset \mathcal{H}$, the desired moment sequence is obtained from (5) by taking $x_0 \in \mathcal{H}$ such that $(1-Q)^{1/2}Qx_0 \neq 0$ and observing that the operator $\widetilde{\rho}(\pi(\widetilde{N}+\widetilde{K}))$ is normal. ∎

3. Some additional results

Recall next (cf. e.g., [8, Chapter IV]) that an operator $T \in \mathcal{L}(\mathcal{H})$ is quasitriangular if there exists a sequence $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ of finite-rank (selfadjoint) projections converging in the SOT to $1_{\mathcal{H}}$ such that

$$\|(1-P_n)TP_n\| \longrightarrow 0,$$

and that T is biquasitriangular if both T and T^* are quasitriangular. For more information about quasitriangular operators, see [8].

The following result is from [1].

THEOREM 3.1: *Associated with every operator T in $\mathcal{L}(\mathcal{H})$ that is not biquasitriangular is a point $\lambda_T \in \sigma(T)$ such that the Fredholm index $i(T-\lambda_T) \neq 0$, and thus T has a nontrivial hyperinvariant subspace.*

COROLLARY 3.2: *Every nonbiquasitriangular operator in $\mathcal{L}(\mathcal{H})$ admits a moment sequence.*

Remark 3.3: It may well turn out (as it does for operators on real separable Banach spaces) that for operators in $\mathcal{L}(\mathcal{H})$, having a moment sequence is equivalent to having a nontrivial invariant subspace. We also note, via the spectral theorem, that *every unitary operator in $\mathcal{L}(\mathcal{H})$ admits a nonzero moment sequence* (supported on the unit circle $\mathbb{T}$). Thus if one changes Definition 1.1 by removing the requirement that the Borel measure be supported on $\sigma(T)$ (let us temporarily call this property "admitting a weak moment sequence"), then it is an easy consequence of the remark above and the fact that every contraction in

$\mathcal{L}(\mathcal{H})$ has a unitary dilation that *every operator in* $\mathcal{L}(\mathcal{H})$ *admits a weak moment sequence*, which is enough said about that concept.

Recall next that an operator $T \in \mathcal{L}(\mathcal{H})$ is called **essentially normal** if $\pi(T)$ is normal in $\mathcal{L}(\mathcal{H})/\mathbf{K}$.

The following is a well known consequence of the Brown–Douglas–Fillmore theorem [5].

THEOREM 3.4: *The set* $(\mathbf{N}+\mathbf{K}) \subset \mathcal{L}(\mathcal{H})$ *is the intersection of the set of biquasitriangular operators and the set of essentially normal operators.*

Thus an immediate corollary of Theorems 1.4, 3.4, and Corollary 3.2 is the following.

COROLLARY 3.5: *Every essentially normal operator in* $\mathcal{L}(\mathcal{H})$ *admits a moment sequence.*

Another easy consequence of Theorem 1.4 is the next result.

THEOREM 3.6 (S. Brown): *Every hyponormal (respectively, cohyponormal) operator* H *in* $\mathcal{L}(\mathcal{H})$ *admits a moment sequence.*

Proof: By Proposition 1.5 it suffices to treat the case in which H is hyponormal. If H has a nontrivial invariant subspace, the result follows from Proposition 1.5. If not, every nonzero vector x in $\mathcal{H}$ is cyclic for H, and by the Berger–Shaw theorem [4], $H \in (\mathbf{N}+\mathbf{K})$, so the result follows from Theorem 1.4. ∎

ACKNOWLEDGEMENT: The first and fourth authors acknowledge the support of the National Science Foundation. The second and third authors were supported by a grant from the Korea Research Foundation (KRF 2002-070-C00006).

References

[1] C. Apostol, C. Foias and D. Voiculescu, *Some results on non-quasitriangular operators*, IV, Revue Roumaine de Mathématiques Pures et Appliquées **18** (1973), 487–514.

[2] W. Arveson, *Subalgebras of C^*-algebras*, Acta Mathematica **123** (1969), 141–224.

[3] A. Atzmon and G. Godefroy, *An application of the smooth variational principle to the existence of nontrivial invariant subspaces*, Comptes Rendus de l'Académie des Sciences, Paris, Série I, Mathématique **332** (2001), 151–156.

[4] C. Berger and B. Shaw, *Selfcommutators of multicyclic hyponormal operators are trace class*, Bulletin of the American Mathematical Society **79** (1973), 1193–1199.

[5] L. Brown, R. G. Douglas and P. Fillmore, *Extensions of C^*-algebras and K-homology,* Annals of Mathematics **105** (1977), 265–324.

[6] S. Brown, *Lomonosov theorem and essentially normal operators,* New Zealand Journal of Mathematics **23** (1994), 11–18.

[7] C. Foias, C. Pasnicu and D. Voiculescu, *Weak limits of almost invariant projections,* Journal of Operator Theory **2** (1979), 79–93.

[8] C. Pearcy, *Some recent developments in operator theory,* CBMS Regional Conference Series in Mathematics, No. 36, Providence, R.I., 1978.

[9] W. Stinespring, *Positive functions on C^*-algebras,* Proceedings of the American Mathematical Society **6** (1955), 211–216.